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Mass Defect in the Noncommutative Schwarzschild Space-Time

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Abstract In this paper we investigate the mass defect and other gravitational effects in noncommutative Schwarzschild space-time obtained by considering particles as smeared objects. The effects of space-time noncommutativity on mass defect of a test particle and a homogeneous spherical shell are calculated. The NC corrections to gravitational redshift, and light-speed in Schwarzschild field are briefly discussed. The results show that the NC corrections have weakening action on these gravitational effects comparing with those in commutative cases.

Keywords Mass defect · Space-time noncommutivity · Gravitational redshift · Light-speed

1 Introduction

Space-time noncommutivity (or NC) is from the concept of a quantized space-time proposed by Snyder [1], and it has played an important role in string theory since 1999 [2]. Noncommutative Minkowski space is defined in terms of space-time coordinates x^μ, $\mu = 0, \ldots, 3$, which satisfy the following commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a real, antisymmetric and constant tensor with very small elements, called noncommutative parameters, which determines the fundamental cell discretization of space-time much in the same way as the Planck constant $\hbar$ discretizes the phase space. It has the dimension of $(length)^2$.

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Gedanken experiments that aim at probing space-time structure at very small distances support the idea that noncommutativity of space-time is a feature of Planck scale physics [3]. Since 1985, noncommutative geometry has arisen as a possible scenario for the short-distances behavior of physical theories. At "large distances" one expects minimal deviations from standard geometry [4]. The occurrence of the effect of NC of both space-space and momentum-momentum is on a very tiny string scale or in a very high energy situation [5]. Noncommutativity is not directly visible at presently accessible energies, i.e. $\theta < 10^{-36}$ m^2 [4]. The NC scale can be probed up to a few TeVs [6]. Many tests have been suggested to detect noncommutative effects since 2001. Due to gravitational back reaction, one cannot test space-time at Planck scale [3]. Most of the tests are imaginative thought experiments or physical applications of its deduction. For example, in ordinary space-time theory, decay of a spin-1 particle into two photons is strictly forbidden due to the Yang's Theorem, but in noncommutative space-time this process can occur [6]. This process thus provides an important probe for noncommutative space-time. Another important kind of these tests is the long-distance noncommutative corrections to many macro physical qualities, e.g., a valuable test of space-time noncommutativity is its possibly observable effects on the properties of black holes [7]. In this paper what we address is some gravitational effects of a test particle in the noncommutative Schwarzschild field.

2 Mass Defection in NC Space-Time

Commutative Schwarzschild geometry is described by the following line element

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \tag{2}$$

where we use the units $G = c = 1$. We want to consider the effect of space-time noncommutativity on this line element. It is possible to consider the effects of space-time noncommutativity on Einstein field equations also. One can argue that it is not necessary to change the Einstein tensor part of the field equations, and that the noncommutative effects can be implemented only on the matter source [4]. Since noncommutativity eliminates point-like objects in favor of smeared objects in flat space-time, we choose the mass density of a static, spherically symmetric smeared, particle-like gravitational source as

$$\rho_\theta(r) = \frac{M}{(4\pi\theta)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{4\theta}\right). \tag{3}$$

This is the theoretical starting point of the coordinate coherent state approach as an important NC quantum field theory. Solving the Einstein equations with this matter source, Nicolini et al. [4] find a deformed Schwarzschild solution. The line element of this NC space-time is

$$\begin{aligned} ds^2 = {} & \left[1 - \frac{2\alpha}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\right]dt^2 - \left[1 - \frac{2\alpha}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\right]^{-1} dr^2 \\ & - r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{4}$$

where $\alpha = 2M$, and $\gamma(3/2, r^2/4\theta)$ is the lower incomplete Gamma function defined as

$$\gamma(3/2, r^2/4\theta) \equiv \int_0^{r^2/4\theta} \sqrt{t} e^{-t} dt. \tag{5}$$

In the limit of $r/\sqrt{4\theta} \to \infty$ the classical Schwarzschild metric is obtained.

In order to discuss gravitational effect in NC Schwarzschild space-time, we first consider the mass defect described as the difference between the inertial rest mass in general relativity theory and the counterpart in Newton theory. Inertial mass equals gravitational mass according to the equivalence principle. Kinetic energy and potential energy could be transformed into inertial mass, and then transformed into gravitational mass. Considering the rest mass of a test particle at finite radius r in the NC Schwarzschild field and in the asymptotic region, we can get the value difference between them [8].

The equation of motion for a test free particle with mass, i.e. the geodesic equation is

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\tau\lambda}\frac{dx^\tau}{ds}\frac{dx^\lambda}{ds} = 0. \tag{6}$$

Taking $\mu = 0$, (6) can be written as

$$\frac{d}{ds}\left(g_{00}\frac{dt}{ds}\right) = 0. \tag{7}$$

And it namely is $g_{00}dt/ds = A$, where A is an integration constant. When $r \to \infty$, the metric becomes Minkowskian, i.e. $dt/ds \to A$. In the theory of special relativity, for a particle of rest mass m_0, the total energy of a particle $E_{\text{total}} = m = m_0 dt/ds$ is deformed as

$$m = m_0 g_{00}\frac{dt}{ds}. \tag{8}$$

Using the line element of the NC Schwarzschild space-time, we can get

$$m = m_0 g_{00}\big(g_{00} + g_{11}\dot{r}^2 + g_{22}\dot{\theta}^2 + g_{33}\dot{\varphi}^2\big)^{-\frac{1}{2}}. \tag{9}$$

As the particle moves along the geodesic to the point (t, r, θ, φ), and rest there. The kinetic energy of this particle is radiated out in the form of heat and other forms of energies. This loss of energy results in the reduction of the rest mass of the particle in the field. And the mass defect is obtained by setting $\dot{r} = \dot{\theta} = \dot{\varphi} = 0$:

$$\Delta m = m' - m_0 = (\sqrt{g_{00}} - 1)m_0, \tag{10}$$

where m' is the rest mass of the test particle which moves from the asymptotic region and rest at finite radius r, m_0 is its inertial rest mass.

Now, plugging g_{00} into (10), we obtain the mass defect of a test particle in the noncommutative Schwarzschild field:

$$\Delta m = \left(\sqrt{1 - \frac{2M}{r} + e^{-\frac{r^2}{4\theta}}\left(\frac{2M}{\sqrt{\pi\theta}}\right)} - 1\right)m_0. \tag{11}$$

From this result, we find that the NC correction to mass defection is determined by the mass of the gravity-source, radius r ($r > 2M$) and NC parameter θ. The third term in the square root in (11) is positive and is extremely small as the θ is very small. Therefore, the absolute value of the modified mass defect is slimly smaller than the commutative case. This indicate that the NC correction has weakening action on gravitational mass defect. This NC weakening action increases as the gravity-source mass M becomes larger, r smaller or θ larger. When the θ vanishes, the mass defect reduces to the case in commutative space-time.

In the following text, we calculate the NC correction to the mass defect of a homogeneous spherical shell. We suppose a and M are, respectively, the radius and mass of the spherical shell, whose thickness is neglected. In the gravitational field induced by spherical shell, we consider a mass unit dM on the surface of it, the mass of the mass unit at infinity is dM_0. Using (11), we obtain

$$dM_0 = \left[1 - \frac{2M}{r} + e^{-\frac{r^2}{4\theta}}\left(\frac{2M}{\sqrt{\pi\theta}}\right)\right]^{-\frac{1}{2}} dM. \tag{12}$$

Integrating (12), we obtain the relationship between M_0 and M

$$M_0 = \int_0^M \left[1 - \frac{2M}{r} + e^{-\frac{r^2}{4\theta}}\left(\frac{2M}{\sqrt{\pi\theta}}\right)\right]^{-\frac{1}{2}} dM. \tag{13}$$

The mass defect of the homogeneous spherical shell is

$$\Delta M = M_0 - M = \int_0^M \left[1 - \frac{2M}{r} + e^{-\frac{r^2}{4\theta}}\left(\frac{2M}{\sqrt{\pi\theta}}\right)\right]^{-\frac{1}{2}} dM - M. \tag{14}$$

The weakening effect exist in it at the same way. The result of (14) after integration is too expatiatory, and on the second thoughts according to actual requirement later.

3 Discussion and Conclusions

The NC correction to the other two gravitational effects are considered. First, we can calculate the gravitational redshift through $\nu/\nu_0 = \sqrt{g_{00}}$, where ν is the frequency of the light infinitely far away from the gravitational resource, and ν_0 is the frequency of the light in the Schwarzschild gravitational space-time. And we introduce the g_{00} in (4) into it, then we obtain a gravitational redshift expression in the noncommutative Schwarzschild space-time:

$$\frac{\nu}{\nu_0} = \sqrt{1 - \frac{2M}{r} + e^{-\frac{r^2}{4\theta}}\left(\frac{2M}{\sqrt{\pi\theta}}\right)}. \tag{15}$$

The modified redshift $z = \nu_0/\nu - 1$ is smaller than the commutative case. This NC correction to gravitational redshift is similar to the foregoing case in Sect. 2, the NC correction to gravitational redshift has weakening action. This is still a good result to analysis the macroscopic effects of the noncommutativity of space-time.

A very accurate gravitational redshift experiment, which deals with this issue, was performed in 1979 [9], where a hydrogen maser clock on a rocket was launched to a height of 10,000 km, and its rate compared with an identical clock on the ground. It tested the gravitational redshift at the 70×10^{-6} level of accuracy, and had deviation 2×10^{-4} from the theoretical value of gravitational redshift, which used the formula $z = \Delta U/c^2$ where ΔU is the gravitational potential between the top site launched and the ground. If the deviation is attributed to the space-time noncommutativity completely, an estimation of NC parameter was given, namely $\theta \approx 9.87 \times 10^{11}\ \mathrm{m}^2$, which is highly bigger than the foregoing upper limit in [4]. This indicates that NC correction to gravitational redshift is only a very tiny part of the deviation, and difficult to be detected by macroscopic effects in current experimental level. But the qualitative character of this result may be significative.

Second, the problem of light-speed is underlying the deflection of starlight and the time delay of light in gravitational field, still is the cornerstone of the theory of Special Relativity and General Relativity. The contribution of the Sun's gravitational field to time delay of electromagnetic echo waves is that the speed of waves is slowed by the field, first known by Shapiro in 1964. The gravitational deflection of light can be looked as a refraction of wavefront. The wavefront near the Sun travel slowly than the wide parts, thus the normal vectors of the wavefront turn to the Sun. The speed of light is one of the fundamental postulates of special relativity. A fundamental change to relativity is needed if c is changing because relativity shows that space and time are equivalent.

From the Schwarzschild metric, we have the speed of light that is travelling radially in a Schwarzschild geometry, $v_r = g_{00}c$, the g_{00} in (4) is introduced into it, then the light-speed in the noncommutative Schwarzschild space-time is obtained:

$$v_r = \left(1 - \frac{4M}{r\sqrt{\pi}}\gamma(3/2, r^2/4\theta)\right)c. \tag{16}$$

The function $\gamma(3/2, r^2/4\theta)$ is positive and rapidly increasedly converges to $\sqrt{\pi}/2$ as $r/\sqrt{\theta}$ approaches infinity. As light gets closer to the center of the gravitating object the coordinate speed of light (v_r) slows down, and the observer is infinitely far away from the object. As a conclusion, the velocity of light in a gravitational field is not a constant, but rather a variable depending upon the site and the reference frame of the observer. Therefore, the light-speed in NC Schwarzschild space-time is larger than the one in commutative case at same position. The NC correction has weakening action on it similarly to the two foregoing results.

As a summary, we have studied three gravitational effects, including mass defect, gravitational redshift and light-speed in NC Schwarzschild geometry. The macroscopic gravitational effects of the space-time noncommutativity which primarily describes the small scale space-time structure are gained. First, the NC corrections to the three gravitational effects are very small due to the tiny NC parameter θ. Although these NC effects are very small, it is important since reflect the nature of space-time structure at quantum gravity level. Second, the NC corrections have weakening action on three gravitational effects. This similarity indicates the NC effects reflect the common properties of space-time, the mechanism of the weakening actions probably reveals the noncommutativity weaken the gravity or the curvature of space-time. All these need further to research combining the quantum gravity.

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